An elementary proof of $\sum_{n\geq 1} 1/n^2 = \pi^2/6$ and a recurrence formula for even zeta values

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Abstract

In this note, a series expansion technique introduced recently by Dancs and He for generating Euler-type formulae for $\zeta(2k+1)$, k being a positive integer and $\zeta(s)$ being the Riemann zeta function, is modified in a manner to furnish the even zeta values $\zeta(2k)$. As a result, I find an elementary proof of $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ and a recurrence formula for $\zeta(2k)$.

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1. Introduction

The Riemann zeta function is defined as $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$, $\Re(s) > 1$. For real values of s, $s > 1$, the convergence of this series is guaranteed by the integral test.[1] For integer even values of s, one has a famous Euler's formula (1740) [4]:

$$\zeta(2k) = \frac{2^{2k-1}\,|B_{2k}|}{(2k)!}\,\pi^{2k}\,, \tag{1}$$

where k is a positive integer and B_k is the k-th Bernoulli number [6]. For instance, since $B_2 = 1/6$ and $B_4 = -1/30$, one finds $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, respectively.

By noting that the approach introduced by Dancs and He in a recent work [3], in which an Euler-type formula is derived for each $\zeta(2k+1)$, can be modified in a manner to yield similar formulas for $\zeta(2k)$, here in this shortnote I show that the substitution of $\sin(n\pi)$ by $\cos(n\pi)$ in the Dancs-He initial series in fact leads to a series expansion involving even zeta values. Since the numbers $E_{2m}(1)$ are null for all positive integer values of m,[2] that series reduces to a finite sum

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[1] For $s = 1$, one has the harmonic series $\sum_{n=1}^{\infty} 1/n$, which diverges to infinity.

[2] Here, $E_m(x) := \sum_{n=0}^{m} \binom{m}{n} \frac{E_n}{2^n} \left(x - \frac{1}{2}\right)^{m-n}$ is the Euler polynomial of degree m.

involving the even zeta values from $\zeta(2)$ to $\zeta(2k)$. By analyzing the first few terms of this finite sum, I have found a new elementary proof for $\zeta(2) = \pi^2/6$, a famous formula first proved by Euler, which is the solution of the so-called Basel problem (see Ref. [2] and references therein), and a recurrence formula for $\zeta(2k)$. My proof is elementary in the sense it does not involve complex analysis, Fourier series, or multiple integrals.[3]

2. An elementary evaluation of $\zeta(2)$

For any real $\epsilon > 0$ and $u \in [1, 1+\epsilon]$, we begin by taking into account the following Taylor series expansion considered by Dancs and He in a previous work [3]:

$$\frac{2\,e^t}{e^t + u} = \sum_{m=0}^{\infty} \phi_m(u)\,\frac{t^m}{m!}\,, \tag{2}$$

which converges absolutely for $|t| < \pi$. From the generating function for $E_m(x)$, namely $2\,e^{x\,t}/(e^t+1) = \sum_{m=0}^{\infty} E_m(x)\,\frac{t^m}{m!}$, it is clear that $\phi_m(1) = E_m(1)$, for all nonnegative integer values of m.

For $u > 1$, we have

$$\phi_m(u) = -2 \sum_{n=1}^{\infty} \frac{n^{\,m}}{(-u)^{\,n}}\,. \tag{3}$$

For $m < 0$, we shall take this series as our definition of $\phi_m(u)$. Therefore

$$\phi_{-m}(1) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^m} = -2\,\zeta^*(m) = 2\,(1 - 2^{1-m})\,\zeta(m) \tag{4}$$

for all integer $m > 1$.

In the same domain, let us define

$$f(u) := \sum_{n=1}^{\infty} \frac{(1/u)^n}{n^2}\,,$$

which will reveal as an useful auxiliary (real) function. Since $\cos(n\pi) = (-1)^n$, let us write $f(u)$ in the form

$$f(u) = \sum_{n=1}^{\infty} (-1)^n\,\frac{\cos(n\,\pi)}{u^n\,n^2}\,. \tag{5}$$

By expanding $\cos(n\pi)$ in a Taylor series, one has

$$f(u) = \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{u^n\,n^2} \cdot \sum_{j=0}^{\infty} (-1)^j\,\frac{(n\pi)^{2j}}{(2j)!} \right] = \sum_{j=0}^{\infty} (-1)^j\,\frac{\pi^{2j}}{(2j)!} \sum_{n=1}^{\infty} (-1)^n\,\frac{n^{2j}}{u^n\,n^2}\,.$$

[3] For proofs involving such non-elementary approaches see, e.g., Refs. [1, 5] and references therein.

By writing this series in terms of $\phi_m(u)$, one has

$$f(u) = \sum_{j=0}^{\infty} (-1)^j \, \frac{\pi^{2j}}{(2j)!} \, \frac{\phi_{2j-2}(u)}{(-2)} \, . \tag{6}$$

We are now in a position to prove the Euler's solution to Basel problem.

Theorem 1 (short evaluation of $\zeta(2)$).

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \, .$$

Proof. By taking the limit as $u \to 1^+$ on both sides of Eq. (6), one has

$$\lim_{u \to 1^+} \sum_{n=1}^{\infty} \frac{1}{u^n \, n^2} = -\frac{1}{2} \, \phi_{(-2)}(1) \, + \frac{1}{2} \, \frac{\pi^2}{2!} \, \phi_0(1) - \frac{1}{2} \, \sum_{j=2}^{\infty} (-1)^j \, \frac{\pi^{2j}}{(2j)!} \, \phi_{2j-2}(1) \, , \tag{7}$$

which, in face of the values of $\phi_{-2}(1)$ stated in Eq. (4), implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2} \, \left[2 \left(1 - 2^{1-2} \right) \zeta(2) \right] + \frac{\pi^2}{4} \, E_0(1) - \frac{1}{2} \, \sum_{j=2}^{\infty} (-1)^j \, \frac{\pi^{2j}}{(2j)!} \, E_{2j-2}(1) \, . \tag{8}$$

Since $E_0(1) = 1$ and $E_m(1) = 0$ for all $m > 0$, the right-hand side of this equation reduces to $-\frac{1}{2} \, \zeta(2) + \pi^2/4$, which implies that

$$\zeta(2) = -\frac{1}{2} \, \zeta(2) + \frac{\pi^2}{4} \, ,$$

and then $\frac{3}{2} \, \zeta(2) = \pi^2/4$.

□

3. Recurrence formula for $\zeta(2k)$

Interestingly, the series expansion presented in the previous section can be easily generalized by just changing the exponent of n from 2 to $2k$. We begin this generalization by defining

$$f_k(u) := \sum_{n=1}^{\infty} \frac{\left(1/u \right)^n}{n^{2k}} \, .$$

Again, since $\cos{(n\pi)} = (-1)^n$, we may write

$$\begin{aligned} f_k(u) \quad &= \quad \sum_{n=1}^{\infty} (-1)^n \, \frac{\cos(n \, \pi)}{u^n \, n^{2k}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{u^n \, n^{2k}} \, \sum_{j=0}^{\infty} (-1)^j \, \frac{(n\pi)^{2j}}{(2j)!} \\ &= \quad \sum_{j=0}^{\infty} (-1)^j \, \frac{\pi^{2j}}{(2j)!} \, \sum_{n=1}^{\infty} (-1)^n \, \frac{n^{2j}}{u^n \, n^{2k}} \, . \end{aligned} \tag{9}$$

On rewriting the last series in terms of $\phi_m(u)$, one finds

$$
\begin{aligned}
f_k(u) &= \sum_{j=0}^{\infty} (-1)^j \frac{\pi^{2j}}{(2j)!} \frac{\phi_{2j-2k}(u)}{(-2)} - \frac{1}{2} \sum_{j=0}^{k-1} (-1)^j \frac{\pi^{2j}}{(2j)!} \phi_{2j-2k}(u) \\
&- \frac{1}{2} \sum_{j=k}^{\infty} (-1)^j \frac{\pi^{2j}}{(2j)!} \phi_{2j-2k}(u) \,. \qquad (10)
\end{aligned}
$$

Now, on substituting $m = j - k$ in the above series, one has

$$
\begin{aligned}
f_k(u) &= -\frac{1}{2} \sum_{m=-k}^{-1} (-1)^{m+k} \frac{\pi^{2m+2k}}{(2m+2k)!} \phi_{2m}(u) - \frac{1}{2} \sum_{m=0}^{\infty} (-1)^{m+k} \frac{\pi^{2m+2k}}{(2m+2k)!} \phi_{2m}(u) \\
&= -\frac{1}{2} (-1)^k \left[\sum_{\widetilde{m}=1}^{k} \frac{(-1)^{\widetilde{m}} \pi^{2k-2\widetilde{m}}}{(2k-2\widetilde{m})!} \phi_{-2\widetilde{m}}(u) + \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m+2k}}{(2m+2k)!} \phi_{2m}(u) \right]. \quad (11)
\end{aligned}
$$

This representation of $f_k(u)$ allows us to prove the following recurrence formula involving the zeta values $\zeta(2k)$.

Theorem 2 (Recurrence formula for $\zeta(2k)$). *For any integer $k > 0$,*

$$
\left(4 - \frac{4}{2^{2k}}\right) \zeta(2k) = \sum_{m=1}^{k-1} \frac{(-1)^{k-m+1}}{(2k-2m)!} \left(2 - \frac{4}{2^{2m}}\right) \pi^{2k-2m} \zeta(2m) - (-1)^k \frac{\pi^{2k}}{(2k)!}.
$$

Proof. By taking the limit as $u \to 1^+$ on both sides of Eq. (11), one has

$$
\lim_{u \to 1^+} \sum_{n=1}^{\infty} \frac{1}{u^n n^{2k}} = -\frac{1}{2} (-1)^k \left[\sum_{m=1}^{k} \frac{(-1)^m \pi^{2k-2m}}{(2k-2m)!} \phi_{-2m}(1) + \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m+2k}}{(2m+2k)!} \phi_{2m}(1) \right]. \quad (12)
$$

From Eq. (4), one knows that $\phi_{-2m}(1) = 2\left(1 - 2^{1-2m}\right)\zeta(2m)$. For nonnegative values of m, one has $\phi_{2m}(1) = E_{2m}(1) = 0$, the only exception being $E_0(1) = 1$. This reduces Eq. (12) to only

$$
\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -(-1)^k \sum_{m=1}^{k} \frac{(-1)^m \pi^{2k-2m}}{(2k-2m)!} \left(1 - 2^{1-2m}\right) \zeta(2m) - (-1)^k \frac{\pi^{2k}}{2\,(2k)!} \,.
$$

By extracting the last term of the sum and isolating $\zeta(2k)$, one finds

$$
\left(2 - \frac{2}{2^{2k}}\right) \zeta(2k) = (-1)^{k+1} \sum_{m=1}^{k-1} \frac{(-1)^m \pi^{2k-2m}}{(2k-2m)!} \left(1 - 2^{1-2m}\right) \zeta(2m) - (-1)^k \frac{\pi^{2k}}{2\,(2k)!} \,.
$$

A multiplication by 2 on both sides yields the desired result.

□

The first few even zeta values can be readily obtained from this recurrence formula. For $k = 1$, the sum in the right-hand side is null and one has

$$3\,\zeta(2) = -(-1)\,\frac{\pi^2}{2}\,,$$

which simplifies to $\zeta(2) = \pi^2/6$, in agreement to our Theorem 1.

For $k = 2$, one has

$$\frac{15}{4}\,\zeta(4) = \frac{\pi^2}{2!}\,\zeta(2) - \frac{\pi^4}{4!}\,.$$

By substituting the value of $\zeta(2)$ found above and multiplying both sides by 4, one finds

$$15\,\zeta(4) = \frac{\pi^4}{3} - \frac{\pi^4}{6} = \frac{\pi^4}{6}\,, \tag{13}$$

which implies that $\zeta(4) = \pi^4/90$.

Note that, by writing the recurrence formula in Theorem 2 in the equivalent form

$$\left(1 - \frac{1}{2^{2k}}\right)\frac{\zeta(2k)}{\pi^{2k}} = \sum_{m=1}^{k-1} \frac{(-1)^{k-m+1}}{(2k-2m)!}\left(\frac{1}{2} - \frac{1}{2^{2m}}\right)\frac{\zeta(2m)}{\pi^{2m}} - \frac{(-1)^k}{4\,(2k)!}\,,$$

it is straightforward to show, by induction on k, that the ratio $\zeta(2k)/\pi^{2k}$ is a rational number for every positive integer k. In fact, to prove this rationality result without making use of Euler's formula, Eq. (1), and/or Bernoulli numbers was my original motivation for exploring the mathematical properties of the Dancs-He series expansions.

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